MathConceptition

2023

S4-S6

Question Booklet

問題簿

Time: 1 hour
Calculators are NOT permitted.

Instructions:

1. **DO NOT OPEN THIS QUESTION BOOKLET UNTIL YOU ARE TOLD TO DO SO.**
2. **If the information printed on your answer sheet is not correct, please inform the invigilator immediately.**
3. **Please use a pencil and write your answers neatly ONLY on the answer sheet provided. DO NOT write or draw in the circle next to each answer box. No mark will be given if you failed to follow this instruction.**
4. **Unless otherwise specified, all answers must be in exact value and in its simplest form. Writing the units for the answers is NOT necessary.**
5. **Rough-work sheets provided will be collected at the end of the contest but they will not be marked.**
6. **Diagrams in this question booklet are not necessarily drawn to scale.**

限時：1 小時
不允許使用計算機。

比賽須知：

1. 未宣布開始前，切勿翻閱此問題簿。

2. 請核對答題紙上列出的資料是否與你相符。如有問題，請舉手。

3. 所有答案必須寫在答題紙內，並須用鉛筆作答。請勿填寫或畫花題號後方的圓圈，否則該題答案將會作廢。

4. 除非題目特別表明，所有答案均不需填寫單位，但必須以準確數值及最簡方式表示。

5. 比賽完結時監考員會收回桌上的草稿紙，但草稿紙上所書寫的任何文字或圖表將不獲評閱。

6. 此問題簿的附圖不一定依比例繪成。

1) Find the slope of $3x + 4y = 36$. [3%]

求 $3x + 4y = 36$ 的斜率。

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2) Let $u(x) = x^2 - 11$. For which non-negative integer value x is $u(x) < -10$? [3.1%]

設 $u(x) = x^2 - 11$。哪個非負整數 x 會使 $u(x) < -10$？

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3) The perimeter of a square is A cm. The area of the square is B cm^2. Given that $A + B = 12$, the side length of the square is ___?___ cm. [3.2%]

某正方形的周界為 A cm。而該正方形的面積為 B cm^2。
已知 $A + B = 12$，該正方形的邊長是 ___?___ cm。

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4) Let $h(x) = k(x - 4)(x + 4)$, where k is a non-zero constant. If $h(a - 3.4) = 0$, for some positive real number a, find the value of a. [3.3%]

設 $h(x) = k(x - 4)(x + 4)$，其中 k 是非零常數。若對於某正實數 a，
$h(a - 3.4) = 0$，求 a 的值。

5) Given that $A = 3^k + 3^{k+2}$, where k is a constant. Express $4(3^k)$ in terms of A. [3.4%]

已知 $A = 3^k + 3^{k+2}$，其中 k 是個常數。試以 A 表示 $4(3^k)$。

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6) On a rectangular coordinate plane, lines L_1 and L_2 are perpendicular to each other. $(0, 2)$ and $(0, s)$ lie on L_1 and L_2 respectively. L_1 and L_2 intersect at $(-1, 0.5)$, find the value of s. [3.5%]

在直角坐標平面上，直線 L_1 和 L_2 互相垂直。$(0, 2)$ 和 $(0, s)$ 分別位於 L_1 和 L_2 之上。L_1 和 L_2 相交於 $(-1, 0.5)$。求 s 的值。

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7) Find the value of the following expression. [3.6%]
求以下數式的值。

$$\cos^2 89° + \cos^2 88° + \cos^2 87° + \cdots + \cos^2 2° + \cos^2 1°$$

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8) There are 5 red balls and 6 blue balls in a bag. Two balls are randomly taken out one by one without replacement. Given that the second ball is red, find the probability that the first ball is blue. [3.7%]

布袋中有 5 個紅色球，6 個藍色球。隨機從中逐個取出兩個球，途中不放回袋中。已知第二個球是紅色，求第一個球是藍色的概率。

9) The coordinates of the lowest point of the graph of $y = f(x)$ are $(-2, \sqrt{3})$. [4.8%]

Find the coordinates of the lowest point of the graph of $y = \dfrac{\sqrt{3}}{3} f(2 - 7x)$.

$y = f(x)$ 的圖像的最低點的坐標是 $(-2, \sqrt{3})$。求 $y = \dfrac{\sqrt{3}}{3} f(2 - 7x)$ 的圖像的最低點的坐標。

10) Solve the following equation, where x is a positive integer. [4.9%]

解以下方程，其中 x 是個正整數。

$$x^{(\log_2 x) - 2} = 256$$

11) Suppose $0 < \theta < 90$, which of the following must be true? [5.1%]

設 $0 < \theta < 90$，下列何者必為正確？

A. $\sin(\sin \theta°)° < 0$

B. $\cos(-\cos \theta°)° < 0$

C. $\sin(\cos \theta°)° < 0$

D. $\cos(\sin \theta°)° < 0$

E. $\sin(\sin \theta°)° - \cos(\sin \theta°)° < 0$

F. $\cos(\cos \theta°)° - \sin(\cos \theta°)° < 0$

12) Figure 1 shows the graph of $y = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$, where a, b, c, d, e and f are constants. Which of the following represents the graph of the second derivative $\dfrac{d^2y}{dx^2}$?   [5.2%]

圖 1 所示為 $y = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$ 的圖像，其中 a、b、c、d、e 和 f 是常數。以下何者為二階導數 $\dfrac{d^2y}{dx^2}$ 的圖像？
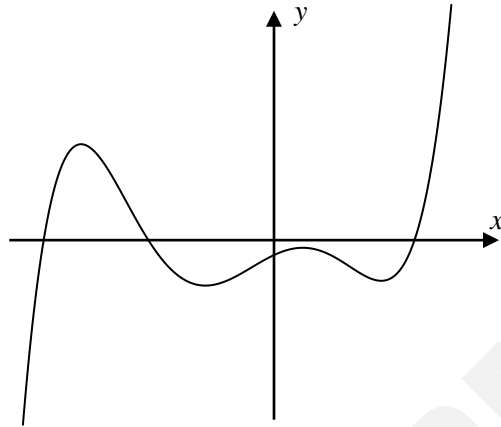

Figure 1　圖 1

A.

B.

C.

D.

E.

F.

G.

H.

I.

13)

> The **Inequality of Arithmetic and Geometric Means (AM-GM Inequality)** states that : [6.3%]
>
> For any real numbers $x_1, x_2, \ldots, x_n \geq 0$,
>
> $$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n} \,,$$
>
> with equality if and only if $x_1 = x_2 = \cdots = x_n$.

Given that $S = 9x^2 + 36x^2y^2 + 4y^2 + z^4$, where x, y and z are positive real numbers and $xyz = \dfrac{1}{6}$. By considering the AM-GM Inequality, or otherwise, find the sum of x, y and z when S attains its minimum value.

> 算術－幾何平均值不等式（算幾不等式）：
>
> 對於任何實數 $x_1, x_2, \ldots, x_n \geq 0$，
>
> $$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n} \,,$$
>
> 等號成立當且僅當 $x_1 = x_2 = \cdots = x_n$。

已知 $S = 9x^2 + 36x^2y^2 + 4y^2 + z^4$，其中 x、y 和 z 為正實數及 $xyz = \dfrac{1}{6}$。藉考慮算幾不等式或利用其他方法，求當 S 達到其最小值時，x、y 及 z 之和。

14) Find the integral part of the following fraction when converted to a decimal. [6.4%]
求當以下的分數被化成小數時的整數部分。

$$\frac{1}{\dfrac{1}{2001}+\dfrac{1}{2002}+\cdots+\dfrac{1}{2022}+\dfrac{1}{2023}}$$

15) For any real number x, $f(x)$ is a function that satisfies $f(x+23)=f(23-x)$. [6.5%]
Given that $f(x)=0$ has 8 unique real roots, find the sum of these 8 unique real roots.

對於任何實數 x，某函數 $f(x)$ 滿足 $f(x+23)=f(23-x)$。已知 $f(x)=0$ 有 8 個不同的實根，求這 8 個實根之和。

16) There are 10 chairs of distinct colours around a circular table. Six gentlemen [6.6%]
and four ladies are to be seated around the table, where each person takes one seat. How many different ways can they be seated, if no two ladies are sitting next to each other?

一張圓桌周圍有 10 張不同顏色的座椅。六位男士和四位女士要圍著該圓桌坐，每人佔一張座椅。若沒有女士相鄰而坐，他們可以有多少種不同的坐法？

17) Consider the general term of a sequence $f_n = n^3 - 4$, $n \geq 1$. Let g_i be the sequence of the Highest Common Factor (H.C.F) of f_i and f_{i+1}. Find the greatest term of g_i. [6.7%]

考慮數列通項 $f_n = n^3 - 4$，$n \geq 1$。設 g_i 為 f_i 和 f_{i+1} 的最大公因數(H.C.F.) 的數列。求數列 g_i 中的最大的項。

18) Consider the arithmetic sequence 11, 23, 35, …, 2399, where the common difference is 12. [6.8%]
Find the last three digits of their product $11 \times 23 \times 35 \times \cdots \times 2399$.

考慮等差數列 11, 23, 35, …, 2399，其公差是 12。
求該數列各項之積 $11 \times 23 \times 35 \times \cdots \times 2399$ 的最後三個數字。

19) In the figure, $\triangle ABC$ is an acute triangle. *BDH* and *CDG* are straight lines [6.9%] and they are the internal angle bisectors of $\angle ABC$ and $\angle ACB$ respectively. Through *D*, a straight line parallel to *BC* is drawn, which meets *AB* and *AC* at *E* and *F* respectively. Given that $AB = 30$, $AC = 48$ and $EF = \dfrac{273}{10}$.

Find the radius of the circumcircle of $\triangle ABC$.

在圖中，$\triangle ABC$ 是一個銳角三角形。*BDH* 和 *CDG* 都是直線，它們分別為內角 $\angle ABC$ 和內角 $\angle ACB$ 的角平分線。一條穿過 *D*，並與 *BC* 平行的線分別與 *AB* 和 *AC* 相交於 *E* 和 *F*。已知 $AB = 30, AC = 48$ 及 $EF = \dfrac{273}{10}$。求 $\triangle ABC$ 的外接圓的半徑。
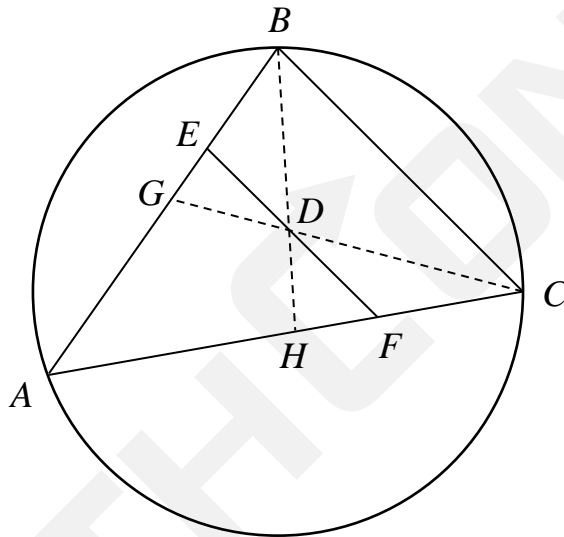
20) Let n be a positive integer and $x_1, x_2, \ldots, x_n$ be real numbers such that [7%]

$$\sum_{i=1}^{n} x_i = 0 \text{ and } \left(\sum_{i=1}^{n} \sum_{j=1}^{n} |x_i - x_j| \right)^2 \le C \sum_{i=1}^{n} x_i^2,$$

Find the minimum possible value of C in terms of n.

設 n 為正整數及 $x_1, x_2, \ldots, x_n$ 為實數使

$$\sum_{i=1}^{n} x_i = 0 \text{ 及 } \left(\sum_{i=1}^{n} \sum_{j=1}^{n} |x_i - x_j| \right)^2 \le C \sum_{i=1}^{n} x_i^2,$$

以 n 表示 C 的最小可能值。

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	ANSWER			ANSWER	
1	$-\dfrac{3}{4}$ or -0.75		11	E	
2	0		12	C	
3	2		13	$\dfrac{11}{6}$ or $1\dfrac{5}{6}$	
4	7.4		14	87	
5	$\dfrac{2}{5}A$ or $0.4A$		15	184	
6	$-\dfrac{1}{6}$		16	432000	
7	$44\dfrac{1}{2}, 44.5, \dfrac{89}{2}$		17	433	
8	$\dfrac{3}{5}$ OR 0.6		18	625	
9	$\left(\dfrac{4}{7}, 1\right)$		19	$14\sqrt{3}$	
10	16		20	$\dfrac{4n(n^2-1)}{3}$	